Indag. Mathem., N.S., 19 (4), 563–578 December, 2008

On metrizability of compactoid sets in non-archimedean locally convex spaces ☆

by Jerzy Kąkol and Wiesław Śliwa

Faculty of Mathematics and Computer Science, A. Mickiewicz University, ul. Umultowska 87, 61-614 Poznań, Poland

Communicated by Prof. M.S. Keane

ABSTRACT

In 2003, N. De Grande-De Kimpe, J. Kąkol and C. Perez-Garcia using t-frames and some machinery concerning tensor products proved that compactoid sets in non-archimedean (LM)-spaces (i.e. the inductive limits of a sequence of non-archimedean metrizable locally convex spaces) are metrizable. In this paper we show a similar result for a large class of non-archimedean locally convex space with a $\mathfrak{L}$-base, i.e. a decreasing base $(U_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ of neighbourhoods of zero. This extends the first mentioned result since every non-archimedean (LM)-space has a $\mathfrak{L}$-base. We also prove that compactoid sets in non-archimedean (DF)-spaces are metrizable.

1. INTRODUCTION

One of the interesting and important problems, both from topology and functional analysis, concerns metrizability of compact sets. Although the Eberlein–Šmulian theorem clarifies the sequential behaviour of weakly compact sets in many locally convex spaces (l.c.s.) (over the field of either real or complex numbers), it gives no information about metrizability of compact sets. There are several particular results dealing with metrizablility of compact sets in concrete classes of l.c.s. H.H. Pfister [21] and M. Valdivia [28] proved metrizability of precompact sets in (DF)-spaces

MSC: 46S10, 46A03, 46A13

Key words and phrases: Compactoid sets, Compactoid resolutions, t-Frames, Non-archimedean (LM)-spaces

☆ This research is supported for the both authors by the project by Ministry of Science and Higher Education, Poland, grant no. N 201 2740 33.

E-mails: kakol@amu.edu.pl (J. Kąkol), sliwa@amu.edu.pl (W. Śliwa).

and dual metric spaces, respectively (by a *dual metric space* we mean the strong dual of a metrizable l.c.s.). The same result has been proved for (LF)-spaces (and then for (LM)-spaces) by B. Cascales and J. Oruhuela [2] and [3].

In [3] B. Cascales and J. Orihuela generalized all mentioned results by showing that in a large class $\mathfrak{G}$ (including all dual metric spaces, (DF)-spaces and (LM)-spaces) every precompact set is metrizable. A short and elementary proof of this result has been shown in [15]. There are however l.c.s. not in class $\mathfrak{G}$ for which the same conclusion holds, for example the spaces $C_p(X)$ of continuous real-valued maps on an uncountable completely regular Hausdorff space X are not in class $\mathfrak{G}$ (see [4]) but for many concrete compact spaces X every compact set in $C_p(X)$ is metrizable, see [1]. In [15] it is provided a general result stating that weakly bounded sets in a l.c.s. E are weakly metrizable iff the strong dual of E is trans-separable. This yields several concrete well-known cases.

The problem becomes more involved when the field $\mathbb{R}$ or $\mathbb{C}$ is replaced by a non-archimedean one $\mathbb{K}$. In l.c.s. over non-archimedean valued fields the precompactness is a restrictive concept. For example if $\mathbb{K}$ is not locally compact, convex precompact sets are reduced to a single point. In the non-archimedean case L. Gruson and M. van der Put introduced the concept of compactoidness, see [23, p. 134]. This notion for non-archimedean metrizable spaces over a locally compact non-archimedean field $\mathbb{K}$ just coincides with the precompactness (see [23, Example 4.S, p. 134]). In [26] W.H. Schikhof proved that an absolutely convex subset A of a l.c.s. E is a metrizable compactoid set if and only if there exists a sequence (x_n) in E which converges to zero and $A \subset \overline{\mathrm{co}}\{x_n\colon n \in \mathbb{N}\}$. C. Perez-Garcia [20] obtained another result of this type in terms of the seminormed space E'_A. She proved that for a l.c.s. with the Hahn–Banach Extension Property a compactoid set A is metrizable if and only if the dual E' endowed with the seminorm $\|f\|_A := \sup\{|f(x)|\colon x \in A\}$ is of countable type.

N. De Grande-De Kimpe, J. Kąkol and C. Perez-Garcia using t-frames and some machinery concerning tensor products proved that compactoid sets in non-archimedean (LM)-spaces are metrizable ([12, Theorem 3.1]; see also [9, Theorem 2.4.3]).

In the present paper we continue this (non-archimedean) line of research and we provide a larger class of non-archimedean l.c.s. for which every compactoid set is metrizable. Developing some ideas of [12] we show that this happens for any l.c.s. E with a $\mathfrak{L}$-base, i.e. a decreasing base $(U_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ of neigbourhoods of zero (Theorem 10). This extends the mentioned result for (LM)-spaces [12, Theorem 3.1], since every (LM)-space has a $\mathcal{L}$-base (Corollary 11). Theorem 10 motivates us to investigate typical stability properties of the class $\mathcal{L}$ of all l.c.s. with a $\mathcal{L}$-base. We show that the class $\mathcal{L}$ is stable by subspaces, countable products, countable inductive and projective limits, and countable direct sums (Propositions 13–15). Since countable projective limits of (LM)-spaces need not be (LM)-spaces, our Theorem 10 essentially improves the main result of [12]. It is not known whether any (DF)-space belongs to the class $\mathcal{L}$. Nevertheless we provide an independent argument showing that compactoid sets in (DF)-spaces are metrizable

(Theorem 19). This provides a non-archimedean counterpart of H.H. Pfister result mentioned above.

Developing some ideas of the proof of [12, Lemma 1.1], we show that compactoid sets in a polar l.c.s. E are metrizable if and only if the space $E'_c = (E', c(E', E))$ is of countable type (Theorem 2). Next, using t-frames (see [13]) we show that a l.c.s. E is of countable type if it has a *compactoid resolution* i.e. an increasing family $(A_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ of compactoid sets such that $\bigcup_{\alpha\in\mathbb{N}^\mathbb{N}} A_\alpha = E$ (Proposition 5). If the sets A_α are only *bounded* we call such resolution *bounded.* Since every metrizable l.c.s. has a bounded resolution (see the proof of Corollary 6), the last fact combined with the previous one yields the following known result [13, Theorem 3.1]: *Every Fréchet–Montel space is of countable type.*

Finally we prove that a metrizable l.c.s. is of countable type if and only if it admits a compactoid resolution if and only if it admits a strong compactoid resolution, i.e. a compactoid resolution $(A_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ such that every compactoid set A in E is contained in A_α for some $\alpha \in \mathbb{N}^\mathbb{N}$ (Theorem 21). This provides a non-archimedean counterpart of a classical Christensen's result [5, Theorem 3.3].

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of Hausdorff locally convex spaces (l.c.s.) and normed spaces we refer to [24], [19] and [23].

2. PRELIMINARIES

We start with some necessary notions and definitions. The field $\mathbb{K}$ is *spherically complete* if any decreasing sequence of closed balls in $\mathbb{K}$ has a non-empty intersection.

Let $B_\mathbb{K}$ denote the set $\{\alpha \in \mathbb{K}\colon |\alpha| \leqslant 1\}$. Let E be a linear space over $\mathbb{K}$.

The linear span of a subset A of E is denoted by $[A]$.

A set $A \subset E$ is *absolutely convex* if for any $\alpha, \beta \in B_\mathbb{K}$ and any $x, y \in A$ we have $\alpha x + \beta y \in A$. If $A \subset E$ then the set

$$\operatorname{co} A = \left\{ \sum_{i=1}^{n} \alpha_i a_i\colon\ n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in B_\mathbb{K}, a_1, \ldots, a_n \in A \right\}$$

is the smallest absolutely convex subset of E that contains A (and it is called an *absolutely convex hull* of A).

A *seminorm* on E is a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leqslant \max\{p(x), p(y)\}$ for all $x, y \in E$.

A seminorm p on E is a *norm* if $\ker p = \{0\}$.

For any seminorm p on E the map $\bar{p} : E_p \to [0, \infty), x + \ker p \to p(x)$ is a norm on $E_p = (E/\ker p)$.

Let E be a l.c.s. over $\mathbb{K}$, i.e. a linear space over $\mathbb{K}$ endowed with a linear Hausdorff topology defined by a family of seminorms on E. The set of all continuous seminorms on E is denoted by $\mathcal{P}(\mathcal{E})$.

We shall say that a l.c.s. E is *of finite type* if for any $p \in \mathcal{P}(E)$ the space E_p is finite-dimensional. A l.c.s. E is said to be *of countable type* if for any $p \in \mathcal{P}(E)$

the normed space $(E_p, \overline{p})$ contains a linearly dense countable subset. Clearly a metrizable l.c.s. E is of countable type if and only if it contains a linearly dense countable subset.

A metrizable complete l.c.s. is called a *Fréchet space*. A normable Fréchet space is called a *Banach space*. Recall that an infinite-dimensional Banach space E of countable type is isomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero with the sup-norm [23].

By an *(LM)-space* we mean a l.c.s. E which is the inductive limit of an inductive sequence (E_n) of metrizable l.c.s.; for details we refer the reader to [9].

The topological dual of a l.c.s. E is denoted by E'. For a subset A of E we set $A^\circ = \{f \in E'\colon\ |f(x)| \leqslant 1 \text{ for } x \in A\}$. For a set $B \subset E'$ of E' we define ${}^\circ B = \{x \in E\colon\ |f(x)| \leqslant 1 \text{ for } f \in B\}$.

A subset A of a l.c.s. E is *polar* if ${}^\circ(A^\circ) = A$. A l.c.s. E is *polar* if for any $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{P}(E)$ with $q \geqslant p$ such that the set $\{x \in E\colon\ q(x) \leqslant 1\}$ is polar. A subset B of a l.c.s. E is *compactoid* if for any neighbourhood U of 0 in E there exists a finite subset S of E such that $B \subset U + \operatorname{co} S$.

A Fréchet space E is said to be a *Fréchet–Montel space* if every bounded subset of E is compactoid.

A set B in a l.c.s. E is *bornivorous* if it absorbs every bounded subset of E. Let $\mathcal{B}(E)$ denote the family of all bounded subsets of a l.c.s. E. The strong dual of a l.c.s. E, that is the topological dual E' of E endowed with the topology $b(E', E)$ of the uniform convergence on bounded subsets of E, is denoted by E'_b. The topological dual E' of a l.c.s. E endowed with the topology $c(E', E)$ of the uniform convergence on compactoid subsets of E (respectively, the weak-star topology $\sigma(E', E)$) is denoted by E'_c (respectively, E'_σ). The weak topology of a l.c.s. E we denote by $\sigma(E, E')$.

Let E and F be l.c.s. The space of all linear continuous maps from E to F is denoted by $L(E, F)$. An operator $T \in L(E, F)$ is an *isomorphism* if T is injective, surjective and the inverse map T^{-1} is continuous. E is *isomorphic* to F ($E \simeq F$) if there exists an isomorphism $T : E \to F$.

3. RESULTS

We start with the following useful fact.

Lemma 1. *Let A be an absolutely convex compactoid set in a polar space E. Then for every $\epsilon > 0$ and every $f \in [A]'$ there exists $\overline{f} \in E'$ such that $|f(a) - \overline{f}(a)| < \epsilon$ for all $a \in A$.*

Proof. Let $\epsilon > 0$ and let $f \in [A]'$. Then for some continuous polar seminorm p on E we have $|f(x)| \leqslant p(x)$ for every $x \in [A]$. Put

$$U = \{x \in E\colon\ p(x) < \epsilon/(\epsilon + 1)\}$$

and let $\alpha \in \mathbb{K}$ with $|\alpha| > 1$. By [16, Theorem 4.1] (or [24, Lemma 8.1]), there exists a finite subset $S \subset \alpha A$ with $A \subset U + \operatorname{co} S$. Clearly, $[S]$ is a finite-dimensional subspace of $[A]$. By [18, Corollary 2.2], there exists $\overline{f} \in E'$ with $\overline{f}|[S] = f|[S]$ such that

$$|\overline{f}(x)| \leqslant (\epsilon + 1)p(x)$$

for all $x \in E$. Let $a \in A$. Then for some $u \in U$ and $d \in [S]$ we have $a = u + d$. Hence $u \in [A]$ and

$$|f(a) - \overline{f}(a)| = |f(u) - \overline{f}(u)| \leqslant (\epsilon + 1)p(u) < \epsilon.$$

Thus $|f(a) - \overline{f}(a)| < \epsilon$ for all $a \in A$. This completes the proof. □

Using Lemma 1 and some ideas from the proof of [12, Lemma 1.1] we shall prove the following theorem.

Theorem 2. *For a polar l.c.s. E the following conditions are equivalent.*

(a) *Every compactoid set in E is metrizable.*
(b) *E'_c is of countable type.*

Proof. (a) $\Rightarrow$ (b). Let $G = E'_c$ and let $p \in \mathcal{P}(G)$. Then for some metrizable absolutely convex compactoid set A in E we have $p \leqslant p_A$, where

$$p_A : G \to [0, \infty), \quad p_A(f) = \sup_{x \in A} |f(x)|.$$

Let $\lambda \in \mathbb{K}$ with $|\lambda| > 1$. By [24, Proposition 8.2], there exists a sequence $(x_n) \subset \lambda A$ with $x_n \to 0$ in E such that A is contained in the closed absolutely convex hull X of the set $\{x_n : n \in \mathbb{N}\}$. Put $H = \ker p_A$. The operator

$$T : G/H \to c_0, \quad T(f + H) = \big(f(x_n)\big)$$

is well defined and linear. For $f \in G$ we have

$$\max_n |f(x_n)| \leqslant |\lambda| p_A(f) \leqslant |\lambda| \sup_{x \in X} |f(x)| = |\lambda| \max_n |f(x_n)|,$$

so

$$\|T(f + H)\|_\infty \leqslant |\lambda| \overline{p_A}(f + H) \leqslant |\lambda| \|T(f + H)\|_\infty.$$

Thus the normed space $(G/H, \overline{p_A})$ is isomorphic to a subspace of the Banach space c_0, so it is of countable type. Let $\{f_n + H : n \in \mathbb{N}\}$ be a linearly dense countable subset of $(G/H, \overline{p_A})$. Then the set $\{f_n + \ker p : n \in \mathbb{N}\}$ is linearly dense in the normed space G_p. It follows that G is of countable type.

(b) $\Rightarrow$ (a). Let A be an absolutely convex compactoid set in E. Let G, p_A, H and $\{f_n + H : n \in \mathbb{N}\}$ be as above. Put $F = [A]$. Then A is a compactoid set in F

([24, Lemma 8.1]) and F is of countable type ([25, Proposition 4.3]). Clearly the functional

$$\|\cdot\|_A : F' \to [0, \infty), \ g \to \sup_{x \in A} |g(x)|$$

is a norm on F'. The set $\{f_n|F\colon n \in \mathbb{N}\}$ is linearly dense in $(F', \|\cdot\|_A)$. Indeed, let $f \in F'$ and $\epsilon > 0$. By Lemma 1 there exists $\overline{f} \in E'$ with $\|f - \overline{f}|F\|_A \leqslant \epsilon$. Moreover, there exists $g \in [\{f_n\colon n \in \mathbb{N}\}]$ with $p_A(\overline{f} - g) \leqslant \epsilon$. Hence $\|f - g|F\|_A \leqslant \epsilon$ and $g|F \in [\{f_n|F\colon n \in \mathbb{N}\}]$.

Thus $(F', \|\cdot\|_A)$ is of countable type. It follows that A is metrizable in E by [12, Lemma 1.1]. Indeed, A is metrizable in $\sigma(F, F')$, by [7, Lemma 2.4]. Using Theorems 4.4 and 5.12 of [24] we obtain that A is metrizable in F, so in E. Hence all compactoid sets in E are metrizable. □

Let E be a polar l.c.s. Since $E_\sigma = (E, \sigma(E, E'))$ is of finite type, it has no subspace isomorphic to c_0; so every bounded set in E_σ is compactoid [11, Corollary 6.7]. It follows that $(E_\sigma)'_c = E'_b$. Thus, using Theorem 2, we get the following known result [24, Theorem 8.3]:

The strong dual E'_b of a polar l.c.s. E is of countable type if and only if every bounded set in E is $\sigma(E, E')$-metrizable.

By Theorem 2 and its proof we get as well the following corollary.

Corollary 3. *For a l.c.s. E the following conditions are equivalent.*

(a) *Every compactoid set in E is metrizable.*
(b) *For every subspace F of countable type in E the space F'_c is of countable type.*

For $\alpha = (\alpha_n), \beta = (\beta_n) \in \mathbb{N}^{\mathbb{N}}$ we write $\alpha \leqslant \beta$ if $\alpha_n \leqslant \beta_n$ for all $n \in \mathbb{N}$. A family $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ of sets is *increasing* if $A_\alpha \subset A_\beta$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ with $\alpha \leqslant \beta$. Similarly we define a *decreasing* family $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ of sets. By a *resolution* of a set A we mean an increasing family $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ of subsets of A with $\bigcup\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\} = A$. A resolution $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ of a l.c.s. E is *compactoid* [*bounded*] if the sets $A_\alpha, \alpha \in \mathbb{N}^{\mathbb{N}}$, are compactoid [bounded] in E.

In order to prove our next result we shall need the following known fact ([17, Lemma 2.1]; see also the proof of [14, Theorem 3.3]); to keep the paper self-contained we give its simple proof.

Lemma 4. *Let $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ be a resolution of an uncountable set A. Then for some $\gamma \in \mathbb{N}^{\mathbb{N}}$ the set A_γ is infinite.*

Proof. We can choose a sequence (k_n) of positive integers such that the set

$$B_n = \bigcup\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}} \text{ and } \alpha_i = k_i \text{ for } 1 \leqslant i \leqslant n\}$$

is uncountable for every $n \in \mathbb{N}$. Let $x_n \in (B_n \setminus \{x_i\colon 1 \leqslant i < n\})$ for $n = 1, 2, 3, \ldots$. Then for every $n \in \mathbb{N}$ there exists $\beta_n = (\beta_{n,i}) \in \mathbb{N}^{\mathbb{N}}$ such that $x_n \in A_{\beta_n}$ and $\beta_{n,i} = k_i$

for $1 \leqslant i \leqslant n$. Let $\gamma_i = \sup_n \beta_{n,i}$ for $i \in \mathbb{N}$. Then $\gamma = (\gamma_i) \in \mathbb{N}^{\mathbb{N}}$ and $\beta_n \leqslant \gamma$ for $n \in \mathbb{N}$. Hence $(x_n) \subset A_\gamma$, so A_γ is infinite. □

We recall the concept of a t-frame for a normed non-archimedean space $E = (E, \|\cdot\|)$, see [13]. Let $t \in (0, 1]$, and let $X \subset (E \setminus \{0\})$. The set X is called a *t-frame* if for each $n \in \mathbb{N}$ and distinct $x_1, x_2, \ldots, x_n \in X$ we have $\mathrm{Vol}(x_1, x_2, \ldots, x_n) \geqslant t^{n-1} \prod_{i=1}^{n} \|x_i\|$, where $\mathrm{Vol}(x_1, x_2, \ldots, x_n) = \|x_1\| \prod_{i=2}^{n} \mathrm{dist}(x_i, [\{x_1, \ldots, x_{i-1}\}])$. It is clear that every t-frame in E is a linearly independent set. Using this concept we get the following useful result (which is a non-archimedean counterpart of [22, Theorem]).

Proposition 5. *A l.c.s. E with a compactoid resolution $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is of countable type.*

Proof. Let $p \in \mathcal{P}(E)$ and let $\pi_p : E \to E_p$ be the quotient map. Put $C_\alpha = \pi_p(A_\alpha)$ for $\alpha \in \mathbb{N}^{\mathbb{N}}$. Then $(C_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a compactoid resolution of the normed space $E_p = (E_p, \overline{p})$. Fix $t \in (0, 1)$. Suppose that E_p contains an uncountable t-frame X; without loss of generality we may assume that $\overline{p}(x) \geqslant 1$ for all $x \in X$. By Lemma 4 the compactoid set $C_\alpha \cap X$ is infinite for some $\alpha \in \mathbb{N}^{\mathbb{N}}$; it is in contradiction with [13, Proposition 2.2]. Thus every t-frame in E_p is countable and we may apply [13, Theorem 2.5] to deduce that the normed space E_p is of countable type. Consequently the whole space E is of countable type. □

Corollary 6. *If every bounded subset of a metrizable l.c.s. E is compactoid, then E is of countable type.*

Proof. Let (U_n) be a base of absolutely convex neighbourhoods of zero in E. Let $\beta \in \mathbb{K}$ with $|\beta| > 1$. For $\alpha = (\alpha_n) \in \mathbb{N}^{\mathbb{N}}$ we put

$$A_\alpha = \bigcap_{n=1}^{\infty} \beta^{\alpha_n} U_n.$$

Clearly, $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a bounded resolution of E; so $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a compactoid resolution. By Proposition 5 the space E is of countable type. □

Last result yields the following well-known fact [13, Theorem 3.1].

Corollary 7. *Every Fréchet–Montel space is of countable type.*

Using [25, Proposition 4.4] we get the following observation.

Remark 8. Let A be a subset of a l.c.s. E. If every countable subset of A is compactoid then A is compactoid.

Indeed, it is easy to see that every countable subset of $\mathrm{co}\, A$ is compactoid. Hence, by [25, Proposition 4.4], the set $\mathrm{co}\, A$ is compactoid.

We will need the following fact which follows from [24, Lemma 10.6] and [25, Proposition 4.5]; we add its short proof.

Lemma 9. *For a l.c.s. E every equicontinuous set A in E' is compactoid in E'_c.*

Proof. There exists a neighbourhood U of zero in E such that $A \subset U^\circ$. The set U° is compactoid in E'_σ by [19, Theorem 4.2]. Applying [24, Lemma 10.6] one gets that the topologies $\sigma(E', E)$ and $c(E', E)$ coincide on U°. Finally using [25, Proposition 4.5] we deduce that the set U° is compactoid in E'_c. $\square$

Making use of Lemma 9, Proposition 5 and Corollary 3 we get our next theorem.

Theorem 10. *Let E be a l.c.s. with a $\mathfrak{L}$-base $(U_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$. Then every compactoid set in E is metrizable.*

Proof. Let F be a subspace of countable type in E. Put $V_\alpha = U_\alpha \cap F$ for $\alpha \in \mathbb{N}^\mathbb{N}$. Clearly, $(V_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$ is a $\mathfrak{L}$-base in F. Then $(V_\alpha^\circ)_{\alpha \in \mathbb{N}^\mathbb{N}}$ is a compactoid resolution of F'_c. By Proposition 5, the space F'_c is of countable type. Corollary 3 completes the proof. $\square$

Last theorem applies to get the following result proved in [12, Theorem 3.1].

Corollary 11. *Every compactoid set in a (LM)-space E is metrizable.*

Proof. Let (E_n) be an inductive sequence of metrizable l.c.s. such that $E = \varinjlim E_n$. Let $(U_{n,k})_{k=1}^\infty$ be a decreasing base of absolutely convex neighbourhoods of zero in E_n for $n \in \mathbb{N}$. Put

$$V_\alpha = \sum_{n=1}^{\infty} U_{n,\alpha_n} \quad \left(= \bigcup_{i=1}^{\infty} \sum_{n=1}^{i} U_{n,\alpha_n} \right)$$

for $\alpha = (\alpha_n) \in \mathbb{N}^\mathbb{N}$. Using [9, Proposition 1.1.7] one gets easily that $(V_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$ is a $\mathfrak{L}$-base in E. Now it is enough to use Theorem 10 to complete the proof. $\square$

Theorem 10 can be also applied to get the following corollary.

Corollary 12. *Let E be a metrizable l.c.s. Then every compactoid set in the strong dual E'_b of E is metrizable.*

Proof. Let $(A_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$ be a bounded resolution of E constructed in the proof of Corollary 6. Then, for every bounded set A in E, there exists $\alpha \in \mathbb{N}^\mathbb{N}$ such that $A \subset A_\alpha$. Thus $(A_\alpha^\circ)_{\alpha \in \mathbb{N}^\mathbb{N}}$ is a $\mathfrak{L}$-base in the space E'_b. Now Theorem 10 applies to complete the proof. $\square$

Denote by $\mathcal{L}$ the family of all locally convex spaces E with a $\mathfrak{L}$-base $(U_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$. Clearly all metrizable l.c.s. belong to $\mathcal{L}$. It is easy to check the following proposition.

Proposition 13. *Let $E \in \mathcal{L}$. Then every subspace F of E belongs to $\mathcal{L}$. If F is a closed subspace of E, then $E/F \in \mathcal{L}$. The completion of E belongs to $\mathcal{L}$.*

We shall show that the class $\mathcal{L}$ is stable also by countable products, locally convex countable inductive and projective limits and locally convex countable direct sums as well.

Proposition 14. *If $(E_n) \subset \mathcal{L}$, then $E = \prod_{n=1}^{\infty} E_n$ belongs to $\mathcal{L}$.*

Proof. Let $(V_\gamma^k)_{\gamma\in\mathbb{N}^\mathbb{N}}$ be a $\mathfrak{L}$-base in E_k, $k \in \mathbb{N}$. Put

$$V_\alpha = \prod_{k=1}^{\alpha_1} V_\alpha^k \times \prod_{k=\alpha_1+1}^{\infty} E_k$$

for $\alpha = (\alpha_n) \in \mathbb{N}^\mathbb{N}$. Clearly, the family $(V_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is decreasing. Let $m \in \mathbb{N}$. Let V_k be a neighbourhood of zero in E_k and let $\beta_k = (\beta_{k,n}) \in \mathbb{N}^\mathbb{N}$ with $V_{\beta_k}^k \subset V_k$ for $1 \leqslant k \leqslant m$. Put

$$\alpha_n = \max\{m, \beta_{1,n}, \dots, \beta_{m,n}\}$$

for $n \in \mathbb{N}$. Then $\alpha = (\alpha_n) \in \mathbb{N}^\mathbb{N}$, $\alpha_1 \geqslant m$ and $\alpha \geqslant \beta_k$ for $1 \leqslant k \leqslant m$. Hence

$$V_\alpha \subset \prod_{k=1}^{m} V_{\beta_k}^k \times \prod_{k=m+1}^{\infty} E_k \subset \prod_{k=1}^{m} V_k \times \prod_{k=m+1}^{\infty} E_k.$$

It follows that $(V_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is a $\mathfrak{L}$-base in E; so $E \in \mathcal{L}$. □

Proposition 15. *If $(E_n) \subset \mathcal{L}$ is an inductive sequence, then $\varinjlim E_n$ belongs to $\mathcal{L}$.*

Proof. Let (N_k) be a partition of $\mathbb{N}$ into infinite subsets and let $\psi_k : \mathbb{N} \to \mathbb{N}$ be strictly increasing with $\psi_k(\mathbb{N}) = N_k$ for $k \in \mathbb{N}$. If $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ with $\alpha \leqslant \beta$, then

$$\alpha \circ \psi_k \leqslant \beta \circ \psi_k$$

for all $k \in \mathbb{N}$. Note that the map $\varphi : \mathbb{N}^\mathbb{N} \to (\mathbb{N}^\mathbb{N})^\mathbb{N}$ defined by

$$\varphi(\alpha) = (\alpha \circ \psi_k)$$

is injective. Moreover, φ is a surjection. Indeed, for $\beta = (\beta_k) \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$ we put

$$\alpha(n) = \beta_k\big(\psi_k^{-1}(n)\big)$$

for $n \in N_k, k \in \mathbb{N}$; then

$$\alpha : \mathbb{N} \to \mathbb{N},\ n \to \alpha(n),$$

and $\varphi(\alpha) = \beta$. Let

$$(V_\gamma^k)_{\gamma \in \mathbb{N}^\mathbb{N}}$$

be a decreasing base of absolutely convex neighbourhoods of zero in E_k for $k \in \mathbb{N}$. For every $\alpha \in \mathbb{N}^\mathbb{N}$ the set

$$V_\alpha = \sum_{k=1}^{\infty} V_{\alpha \circ \psi_k}^k$$

is a neighbourhood of zero in E [9, Proposition 1.1.7]. We claim that $(V_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$ is a $\mathfrak{L}$-base in $E = \varinjlim E_n$. Indeed, let V be a neighbourhood of zero in E. Then for every $k \in \mathbb{N}$ there exists a neighbourhood V_k of zero in E_k such that

$$\sum_{k=1}^{\infty} V_k \subset V$$

by [9, Proposition 1.1.7]. Let $\beta_k \in \mathbb{N}^\mathbb{N}$ with $V_{\beta_k}^k \subset V_k, k \in \mathbb{N}$. Then $\beta = (\beta_k) \in (\mathbb{N}^\mathbb{N})^\mathbb{N}$, so $\beta = \varphi(\alpha)$ for some $\alpha \in \mathbb{N}^\mathbb{N}$. Hence

$$V_\alpha = \sum_{k=1}^{\infty} V_{\beta_k}^k \subset \sum_{k=1}^{\infty} V_k \subset V.$$

Thus $(V_\alpha)_{\alpha \in \mathbb{N}^\mathbb{N}}$ is a $\mathfrak{L}$-base in E, so we proved that $E \in \mathcal{L}$. $\square$

Last Proposition 15 easily yields the following corollary.

Corollary 16. *If $(E_n) \subset \mathcal{L}$, then the locally convex direct sum $E = \bigoplus_{n=1}^{\infty} E_n$ belongs to $\mathcal{L}$ as well.*

The Propositions 13 and 14 yield also the corollary.

Corollary 17. *If $(E_n) \subset \mathcal{L}$ is a projective sequence of l.c.s., then the projective limit $\varprojlim E_n$ belongs to $\mathcal{L}$.*

It seems to be very likely that (DF)-spaces need not belong to the class $\mathcal{L}$. Nevertheless, we prove also that compactoid sets in (DF)-spaces are metrizable. Recall that a l.c.s. E is a *(DF)-space* if it has a fundamental sequence (B_n) of bounded sets and for every sequence (V_n) of absolutely convex neighbourhoods of zero in E such that the set $V = \bigcap_{n=1}^{\infty} V_n$ is bornivorous, V is a neighbourhood of zero in E.

We shall need the following additional facts which use some tensor machinery. By $\tilde{\mathbb{K}} = (\tilde{\mathbb{K}}, |\cdot|)$ we denote the spherical completion of $\mathbb{K}$ constructed in [23].

Let E be a l.c.s. over $\mathbb{K}$. We equip the tensor product $H = \tilde{\mathbb{K}} \otimes E$ with the locally $\mathbb{K}$-convex topology π induced by the $\mathbb{K}$-seminorms $|\cdot| \otimes p$, where $p \in \mathcal{P}(E)$ and

$$(|\cdot| \otimes p)(z) = \inf\left\{ \max_{1\leqslant i\leqslant n} |\alpha_i| p(x_i)\colon\ n \in \mathbb{N} \text{ and } z = \sum_{i=1}^{n} \alpha_i \otimes x_i \right\}$$

for all $z \in H$, see [9, 2.4.3]. We can consider H as a $\tilde{\mathbb{K}}$-linear space by extending the multiplication by the formula

$$\tilde{\mathbb{K}} \times H \to H, \qquad \left(\alpha, \sum_{i=1}^{n} \alpha_i \otimes x_i\right) \to \sum_{i=1}^{n} (\alpha\alpha_i) \otimes x_i.$$

It is easy to check that $|\cdot| \otimes p$ is a $\tilde{\mathbb{K}}$-seminorm on H for every $p \in \mathcal{P}(E)$. Thus the topology π is locally $\tilde{\mathbb{K}}$-convex. The sets $\operatorname{co}(U \otimes V)$, where U and V are neighbourhoods of zero in $\tilde{\mathbb{K}}$ and E, respectively, form a base of neighbourhoods of zero in H. The map

$$\Phi\colon E \to H,\ x \to 1 \otimes x,$$

is $\mathbb{K}$-linear and it is a homeomorphism into.

Lemma 18. *If E is a (DF)-space over the field $\mathbb{K}$, then the space $H = \tilde{\mathbb{K}} \otimes E$ is a (DF)-space over $\tilde{\mathbb{K}}$.*

Proof. Put $B = \{\alpha \in \tilde{\mathbb{K}}\colon\ |\alpha| \leqslant 1\}$ and let (B_n) be a fundamental sequence of bounded sets in E. By [9, 2.4.3.(III)], every bounded set in H is contained in $\operatorname{co}(B \otimes W)$ for some bounded set W in E. Thus $(\operatorname{co}(B \otimes B_n))$ is a fundamental sequence of bounded sets in H.

Let (U_n) be a sequence of absolutely convex neighbourhoods of zero in H such that the set $U = \bigcap_{n=1}^{\infty} U_n$ is bornivorous in H. Clearly, $V_n = \Phi^{-1}(U_n)$ is an absolutely convex neighbourhood of zero in E for every $n \in \mathbb{N}$. The set $V = \bigcap_{n=1}^{\infty} V_n$ is bornivorous in E. Indeed, let A be a bounded set in E. Then $B \otimes A$ is bounded in H so $B \otimes A \subset \alpha U$ for some $\alpha \in \tilde{\mathbb{K}}$. Let $\beta \in \mathbb{K}$ with $|\beta| \geqslant |\alpha|$. Thus $B \otimes A \subset \beta U$ and

$$A \subset \Phi^{-1}(B \otimes A) \subset \beta\Phi^{-1}(U) = \beta V.$$

Since E is a (DF)-space, so V is a neighbourhood of zero in E. Hence $\operatorname{co}(B \otimes V)$ is a neighbourhood of zero in H. Moreover,

$$1 \otimes V \subset \bigcap_{n=1}^{\infty} 1 \otimes V_n \subset \bigcap_{n=1}^{\infty} U_n = U.$$

For all $\alpha \in B$ and $x \in V$ we have

$$\alpha \otimes x = \alpha(1 \otimes x) \in \alpha U \subset U.$$

Thus

$$\operatorname{co}(B \otimes V) \subset U,$$

so U is a neighbourhood of zero in H. Thus H is a (DF)-space over $\tilde{\mathbb{K}}$. $\square$

Theorem 19. *Every compactoid set in a (DF)-space E is metrizable.*

Proof. First we show that every countable bounded subset $F = \{f_n\colon n \in \mathbb{N}\}$ of E'_b is equicontinuous. Put $F_n = \{f_i\colon 1 \leqslant i \leqslant n\}$ for $n \in \mathbb{N}$. Then ${}^\circ F = \bigcap_{n=1}^{\infty} {}^\circ F_n$. Clearly

$${}^\circ F_n = \bigcap_{k=1}^{n} f_k^{-1}(B_{\mathbb{K}})$$

is an absolutely convex neighbourhood of zero in E for $n \in \mathbb{N}$. Moreover, ${}^\circ F$ is a bornivorous set in E. Indeed, let B be a bounded set in E. Then for some $\alpha \in \mathbb{K}$ we have $F \subset \alpha B^\circ$, so

$${}^\circ F \supset \alpha^{-1}\big({}^\circ(B^\circ)\big) \supset \alpha^{-1} B.$$

Thus ${}^\circ F$ is a neighbourhood of zero in E, so F is equicontinuous. Now we consider two cases:

(1) *E is polar.* Let (B_n) be a fundamental sequence of bounded sets in E and let $\beta \in \mathbb{K}$ with $|\beta| > 1$. Set

$$A_\alpha = \bigcap_{k=1}^{\infty} \beta^{\alpha_k} B_k^\circ$$

for $\alpha = (\alpha_k) \in \mathbb{N}^{\mathbb{N}}$. Clearly $(A_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a bounded resolution of E'_b. By Lemma 9, every countable subset of A_α, $\alpha \in \mathbb{N}^{\mathbb{N}}$, is compactoid in E'_c. Thus, by Remark 8, E'_c has a compactoid resolution. Using Proposition 5 we deduce that E'_c is of countable type. It follows that compactoid sets in E are metrizable.

(2) *E is not polar.* Then $\mathbb{K}$ is not spherically complete. By Lemma 18, $H = \tilde{\mathbb{K}} \otimes E$ is a polar (DF)-space over the field $\tilde{\mathbb{K}}$. Let A be a compactoid set in E. Then $1 \otimes A$ is a compactoid set in H. By (1), $1 \otimes A$ is a metrizable subset of H. Hence A is metrizable, because it is homeomorphic to $1 \otimes A$. $\square$

By the proof of Theorem 19 we get the following remark.

Remark 20. For any polar (DF)-space E the dual E'_c has a compactoid resolution.

Finally, in connection to Proposition 5 we shall prove that any metrizable l.c.s. of countable type has a compactoid resolution.

We shall say that a compactoid resolution $(S_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ of a l.c.s. E is *strong* if every compactoid set in E is contained in some S_α. Clearly every hemicompacoid space (i.e. a l.c.s. having a fundamental sequence (S_n) of compactoid sets) has a strong compactoid resolution. Recall also that a separable metric space X admits a strong compact resolution if and only if it is complete (see [5, Theorem 3.3]).

Theorem 21. *Every metrizable l.c.s. E of countable type has a strong compactoid resolution $(S_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$.*

Proof. (A). First we show that c_0 has a compactoid resolution. By [8, Corollary 3.7], a subset A of c_0 is compactoid if and only if there exists $(\gamma_k) \in c_0$ such that for all $(x_n) \in A$ and $k \in \mathbb{N}$ we have $|x_k| \leqslant |\gamma_k|$.

(A1). Assume that $(\mathbb{K}, |\cdot|) = (\mathbb{Q}_p, |\cdot|_p)$ for some prime number p. Then c_0 is separable. By [27, Theorem 3], there exists a homeomorphism $\psi : \mathbb{N}^\mathbb{N} \to c_0$. Let

$$A_\alpha = \psi\big(\{\beta \in \mathbb{N}^\mathbb{N}\colon \ \beta_n \leqslant \alpha_n \ \text{ for all } \ n \in \mathbb{N}\}\big)$$

for $\alpha = (\alpha_n) \in \mathbb{N}^\mathbb{N}$; A_α is compact, so it is compactoid in c_0. Clearly, $A_\alpha \subset A_\beta$, if $\alpha \leqslant \beta$; thus $(A_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is a compactoid resolution of c_0. Let $\alpha \in \mathbb{N}^\mathbb{N}$. Then there exists $t_\alpha = (t_{\alpha,n}) \in c_0$ such that $|x_n|_p \leqslant |t_{\alpha,n}|_p$ for all $x = (x_j) \in A_\alpha$, $n \in \mathbb{N}$. Let

$$s_{\alpha,n} = \sup\{|x_n|_p\colon \ x = (x_j) \in A_\alpha\}$$

for $n \in \mathbb{N}$. Then $s_{\alpha,n} \leqslant |t_{\alpha,n}|_p$ for $n \in \mathbb{N}$, so $s_{\alpha,n} \to_n 0$. If $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ and $\alpha \leqslant \beta$, then $s_{\alpha,n} \leqslant s_{\beta,n}$ for $n \in \mathbb{N}$.

(A2). Now let $\mathbb{K}$ be arbitrary. Let

$$B_\alpha = \{(y_n) \in c_0\colon \ |y_n| \leqslant s_{\alpha,n} \ \text{for} \ \ n \in \mathbb{N}\}$$

for $\alpha \in \mathbb{N}^\mathbb{N}$. Clearly, $B_\alpha \subset B_\beta$ if $\alpha \leqslant \beta$. Let $(y_n) \in c_0$ and $\lambda \in \mathbb{Q}_p$ with $|\lambda|_p > 1$. Thus there exists $(x_n) \subset \mathbb{Q}_p$ such that

$$|y_n| \leqslant |x_n|_p \leqslant |\lambda|_p|y_n|$$

for $n \in \mathbb{N}$. Hence $|x_n|_p \to 0$, so $(x_n) \in A_\alpha$ for some $\alpha \in \mathbb{N}^\mathbb{N}$. Then $(y_n) \in B_\alpha$. It follows that $(B_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is a resolution of c_0. Let $\gamma \in \mathbb{K}$ with $|\gamma| > 1$. Let $\alpha \in \mathbb{N}^\mathbb{N}$. Then there exists $(\psi_{\alpha,n}) \subset \mathbb{K}$ with

$$s_{\alpha,n} \leqslant |\psi_{\alpha,n}| \leqslant |\gamma| s_{\alpha,n}$$

for $n \in \mathbb{N}$. Hence $(\psi_{\alpha,n}) \in c_0$ and $|y_n| \leqslant |\psi_{\alpha,n}|$ for all $(y_j) \in B_\alpha$, $n \in \mathbb{N}$. Thus B_α is compactoid in c_0. Let B be a compactoid set in c_0. Then there exists $(z_n) \in c_0$ such that $|y_n| \leqslant |z_n|$ for all $(y_j) \in B$, $n \in \mathbb{N}$. For some $\alpha \in \mathbb{N}^\mathbb{N}$ we have $(z_n) \in B_\alpha$. Hence $|z_n| \leqslant s_{\alpha,n}$ for $n \in \mathbb{N}$. It follows that $B \subset B_\alpha$. We have shown that $(B_\alpha)_{\alpha\in\mathbb{N}^\mathbb{N}}$ is a strong compactoid resolution of c_0.

(B). Now we show that $c_0^\mathbb{N}$ has a strong compactoid resolution. Let

$$\varphi : \mathbb{N}^\mathbb{N} \to \left(\mathbb{N}^\mathbb{N}\right)^\mathbb{N}$$

be the bijection constructed in the proof of Proposition 15. For $\alpha \in \mathbb{N}^\mathbb{N}$ we put $D_\alpha = \prod_{n=1}^\infty B_{\alpha^n}$, where $(\alpha^n) = \varphi(\alpha)$; clearly $D_\alpha \subset D_\beta$ if $\alpha \leqslant \beta$.

Let D be a compactoid set in $c_0^{\mathbb{N}}$. Let

$$\pi_k : c_0^{\mathbb{N}} \to c_0, \ (\psi_n) \to \psi_k$$

for $k \in \mathbb{N}$. Then $D_k = \pi_k(D)$ is a compactoid set in c_0 and $D \subset \prod_{k=1}^{\infty} D_k$. For every $k \in \mathbb{N}$ there exists $\gamma^k \in \mathbb{N}^{\mathbb{N}}$ such that $D_k \subset B_{\gamma^k}$. Let $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $\varphi(\alpha) = (\gamma^k)$. Then

$$D \subset \prod_{k=1}^{\infty} B_{\gamma^k} = D_\alpha.$$

It follows that

$$\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} D_\alpha = c_0^{\mathbb{N}}.$$

By [6, Proposition 1.7] we get that $(D_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a strong compactoid resolution of $c_0^{\mathbb{N}}$.

(C). Let F be a subspace of $c_0^{\mathbb{N}}$. Then

$$K_\alpha = D_\alpha \cap F, \quad \alpha \in \mathbb{N}^{\mathbb{N}},$$

form a strong compactoid resolution of F. By [10, Remark 3.6], E is isomorphic to a subspace of $c_0^{\mathbb{N}}$, so it has a strong compactoid resolution $(S_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$. $\square$

Remark 22. If the field $\mathbb{K}$ is locally compact, the proof of Theorem 21 goes easier.

Indeed, in this case E is separable and every compactoid set in E is precompact. Let F be the completion of E. For a dense sequence (x_n) in F and $\alpha = (\alpha_k) \in \mathbb{N}^{\mathbb{N}}$ we put

$$K_\alpha := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\alpha_k} B(x_j, k^{-1}),$$

where $B(x_j, k^{-1})$ is the closed ball in the metric space F with the center at point x_j and radius k^{-1}. It is not hard to check that $(K_\alpha)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a strong compact resolution of F. Consequently $(K_\alpha \cap E)_{\alpha \in \mathbb{N}^{\mathbb{N}}}$ is a strong compactoid resolution of E.

Proposition 5 combined with Theorem 21 yields the following characterization of metrizable l.c.s. of countable type.

Corollary 23. *For a metrizable l.c.s. E the following assertions are equivalent.*

(i) *E is of countable type.*
(ii) *E admits a compactoid resolution.*
(iii) *E admits a strong compactoid resolution.*

ACKNOWLEDGEMENT

The authors wish to thank the referee for Remark 8.

REFERENCES

[1] Arkhangel'skii A.V. – Topological Function Spaces, Kluwer Academic Publisher, Dordrecht, 1992.
[2] Cascales B., Orihuela J. – Metrizability of precompact subsets in (LF)-space, Proc. Edinburgh Math. Soc. **103** (1986) 293–299.
[3] Cascales B., Orihuela J. – On compactness in locally convex spaces, Math. Z. **195** (1987) 365–381.
[4] Cascales B., Kąkol J., Saxon S.A. – Metrizability vs. Fréchet–Urysohn property, Proc. Amer. Math. Soc. **131** (2003) 3623–3631.
[5] Christensen J.P.R. – Topology and Borel Structure, North-Holland, Amsterdam, 1974.
[6] De Grande-De Kimpe N. – The non-archimedean space $C^{\infty}(X)$, Compositio Math. **48** (1983) 297–309.
[7] De Grande-De Kimpe N. – Non-archimedean topologies of countable type and associated operators, Indag. Mathem. **49** (1987) 15–27.
[8] De Grande-De Kimpe N. – Nuclear topologies on non-archimedean locally convex spaces, Indag. Mathem. **49** (1987) 279–292.
[9] De Grande-De Kimpe N., Kąkol J., Perez-Garcia C., Schikhof W.H. – p-adic locally convex inductive limits, in: p-adic Functional Analysis (Nijmegen, 1996), Lecture Notes in Pure and Appl. Math., Dekker, New York, 1997, pp. 159–222.
[10] De Grande-De Kimpe N., Kąkol J., Perez-Garcia C., Schikhof W.H. – Orthogonal sequences in non-archimedean locally convex spaces, Indag. Mathem. (N.S.) **11** (2000) 187–195.
[11] De Grande-De Kimpe N., Kąkol J., Perez-Garcia C., Schikhof W.H. – Orthogonal and Schauder bases in non-archimedean locally convex spaces, in: p-adic Functional Analysis (Ioannina, 2000), Lecture Notes in Pure and Appl. Math., Dekker, New York, 2001, pp. 103–126.
[12] De Grande-De Kimpe N., Kąkol J., Perez-Grcia C. – Metrizability of compactoid sets in non-archimedean Hausdorff (LM)-spaces, Contemporary Math. (Ultrametric Functional Analysis) **319** (2003) 99–107.
[13] De Grande-De Kimpe N., Perez-Garcia C., Schikhof W.H. – Non-archimedean t-frames and FM-spaces, Canad. Math. Bull. **35** (1992) 475–483.
[14] Drewnowski L. – Resolutions of topological linear spaces and continuity of linear maps, J. Math. Anal. Appl. **335** (2007) 1177–1195.
[15] Ferrando J.C., Kąkol J., López Pellicer M. – Necessary and sufficient conditions for precompact sets to be metrizable, Bull. Austr. Math. Soc. **74** (2006) 7–13.
[16] Katsaras A.K. – On compact operators between non-archimedean spaces, Annales Soc. Scientifique Bruxelles **96** (1982) 129–137.
[17] Mercourakis S., Stamati E. – A new class of weakly K-analytic Banach spaces, Comment. Math. Univ. Carolin **47** (2006) 291–312.
[18] Perez-Garcia C. – Semi-Fredholm operators and the Calkin algebra in p-adic analysis I–II, Bull. Soc. Math. Belg. **42** (1990) 69–101.
[19] Perez-Garcia C. – Locally convex spaces over non-archimedean valued fields, Contemporary Math. (Ultrametric Functional Analysis) **319** (2003) 251–279.
[20] Perez-Garcia C. – Compactoid sets in p-adic locally convex spaces with the Hahn–Banach Extension Property, Annales de la Soc. Scien. de Bruxelles **61** (1988) 97–104.
[21] Pfister H.H. – Bemerkungen zum Satz über die separabilität der Fréchet–Montel Raüme, Arch. Math. (Basel) **27** (1976) 86–92.
[22] Robertson N. – The metrisability of precompact sets, Bull. Austral. Math. Soc. **43** (1991) 131–135.
[23] van Rooij A.C.M. – Non-archimedean Functional Analysis, Marcel Dekker, New York, 1978.
[24] Schikhof W.H. – Locally convex spaces over non-spherically complete valued fields I–II, Bull. Soc. Math. Belgique **38** (1986) 187–224.
[25] Schikhof W.H. – Topological stability of p-adic compactoids under continuous injections, Report 8644, Department of Mathematics, University of Nijmegen, The Netherlands, 1986, pp. 1–21.

[26] Schikhof W.H. – A connection between p-adic Banach spaces and locally convex compactoids, Report 8736, Department of Math. Catholic University, Nijmegen, The Netherlands, 1987, pp. 1–16.
[27] Śliwa W. – On topological classification of non-archimedean Fréchet spaces, Czechoslovak Math. J. **54** (2004) 457–463.
[28] Valdivia M. – Topics in Locally Convex Spaces, Notas de Matemática, North-Holland, Amsterdam, 1982.

(Received May 2008)